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3-D Rigid Motion
3-D Rigid Motion

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = R \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + T = \begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

Rotation matrix (9 unknowns)  Translation (3 unknowns)
Rotation

\[ X = R \cos \phi \]
\[ Y = R \sin \phi \]

\[ X' = R \cos(\Theta + \phi) = R \cos \Theta \cos \phi - R \sin \Theta \sin \phi \]
\[ Y' = R \sin(\Theta + \phi) = R \sin \Theta \cos \phi + R \cos \Theta \sin \phi \]

\[ X' = X \cos \Theta - Y \sin \Theta \]
\[ Y' = X \sin \Theta + Y \cos \Theta \]

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]
Rotation (continued)

\[
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta \\
\end{bmatrix}
\]
Euler Angles

\[ R = R_z^\alpha R_y^\beta R_x^\gamma = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix} \]

\[ R = R_z^\alpha R_y^\beta R_x^\gamma = \begin{bmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma \\ -\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma \end{bmatrix} \]

If angles are small: \[ \cos \Theta \approx 1 \quad \sin \Theta \approx \Theta \]

\[ R = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} \]
Image Motion Models
Displacement Model
Image Formation: Orthographic Projection

Image Plane

World point

\( (X, Y, Z) \)

image

Orthographic Projection

\( (X, Y, Z) = (x, y) \)
Orthographic Projection

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = R \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + T = \begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

\[
x = X
\]
\[
y = Y
\]
\[
x' = r_{11}X + r_{12}Y + (r_{13}Z + T_X)
\]
\[
y' = r_{21}X + r_{22}Y + (r_{23}Z + T_Y)
\]

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
a_1 & a_2 \\
a_3 & a_4
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

x' = Ax + b

(x,y)=image coordinates, (X,Y,Z)=world coordinates

Affine Transformation
Orthographic Projection (contd.)

\[
\begin{bmatrix}
X' \\
Y \\
Z'
\end{bmatrix} = R \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + T = \begin{bmatrix}
1 & -\alpha & \beta \\
\alpha & 1 & -\gamma \\
-\beta & \gamma & 1
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

\[
x' = x - \alpha y + \beta Z + T_X \\
y' = \alpha x + y - \gamma Z + T_Y
\]
Image Formation: Perspective Projection

\[ -\frac{y}{Y} = \frac{f}{Z} \]

\[ Y = \frac{fY}{Z} \]

\[ y = -\frac{fY}{Z} \]

\[ x = -\frac{fX}{Z} \]
Perspective Projection

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = R \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + T = \begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

\[
X' = r_{11}X + r_{12}Y + r_{13}Z + T_X \\
Y' = r_{21}X + r_{22}Y + r_{23}Z + T_Y \\
Z' = r_{31}X + r_{32}Y + r_{33}Z + T_Z
\]

\[
x' = \frac{r_{11}x + r_{12}y + r_{13} + \frac{T_X}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}} \\
y' = \frac{r_{21}x + r_{22}y + r_{23} + \frac{T_Y}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}}
\]

scale ambiguity

focal length = -1

\[
x' = \frac{X'}{Z'} \\
y' = \frac{Y'}{Z'}
\]
Plane+Perspective (projective)

$$aX + bY + cZ = 1$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 1$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$A = R + T \begin{bmatrix} a & b & c \end{bmatrix}$$

3d rigid motion
Plane+Perspective (projective)

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = A
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

\[
x' = \frac{X'}{Z'} \quad y' = \frac{Y'}{Z'}
\]

focal length = -1

\[
x' = \frac{a_1X + a_2Y + a_3Z}{a_7X + a_8Y + a_9Z}
\]

\[
y' = \frac{a_4X + a_5Y + a_6Z}{a_7X + a_8Y + a_9Z}
\]

\[
X' = a_1X + a_2Y + a_3Z
\]

\[
Y' = a_4X + a_5Y + a_6Z
\]

\[
Z' = a_7X + a_8Y + a_9Z
\]

\[
a_9 = 1
\]

scale ambiguity
Plane+perspective (contd.)

\[
x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}
\]
\[
y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}
\]

\[
X' = \frac{A X + b}{C^T X + 1}
\]

\[
X' = \begin{bmatrix} x' \\ y' \end{bmatrix}, A = \begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}
\]

\[
b = \begin{bmatrix} a_3 \\ a_6 \end{bmatrix}, C = \begin{bmatrix} a_7 \\ a_8 \end{bmatrix},
\]

Projective
Or
Homography
Least Squares

• Eq of a line

\[ mx + c = y \]

• Consider n points

\[ mx_1 + c = y_1 \]
\[ \vdots \]
\[ mx_n + c = y_n \]

\[ \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \]

\[ Ap = Y \]
Least Squares Fit

\[ A p = Y \]

\[ A^T A p = A^T Y \]

\[ p = (A^T A)^{-1} A^T Y \]

\[ \min \sum_{i=1}^{n} (y_i - mx_i - c)^2 \]
Determining Projective transformation using point correspondences
Determining Projective transformation using point correspondences

- If point correspondences \((x,y)\leftarrow\rightarrow(x',y')\) are known
- \(a\)'s can be determined by least squares fit

\[
x' = \frac{a_1x + a_2y + a_3}{a_7x + a_8y + 1}
\]
\[
y' = \frac{a_4x + a_5y + a_6}{a_7x + a_8y + 1}
\]

\[
a_7x'x + a_8x'y + x' = a_1x + a_2y + a_3
\]
\[
a_7y'x + a_8y'y + y' = a_7x + a_8y + a_6
\]
\[
x' = a_1x + a_2y + a_3 - a_7x'x - a_8x'y
\]
\[
y' = a_7x + a_8y + a_6 - a_7y'x - a_8y'y
\]
\[
a_1x + a_2y + a_3 - a_7x'x - a_8x'y = x'
\]
\[
a_7x + a_8y + a_6 - a_7y'x - a_8y'y = y'
\]

Two rows for each point \(i\)

\[
\begin{bmatrix}
  x_i & y_i & 1 & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \\
  0 & 0 & x_i & y_i & 1 & -x_i y'_i & -y_i y'_i \\
  \vdots & & & & & \vdots \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8 \\
\end{bmatrix}
= \begin{bmatrix}
x'_i \\
y'_i \\
\vdots \\
\end{bmatrix}
\]
Determining Projective transformation using point correspondences

\[
\begin{bmatrix}
  x_i & y_i & 1 & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \\
  0 & 0 & 0 & x_i & y_i & 1 & -x_i y'_i & -y_i y'_i \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7 \\
  a_8 \\
\end{bmatrix}
= \begin{bmatrix}
  \vdots \\
  x'_i \\
  y'_i \\
  \vdots \\
\end{bmatrix}
\]

\[
A a = x'
\]

\[
a = (A^T A)^{-1} A^T x'
\]
Summary of Displacement Models

Translation
\[
x' = x + b_1 \\
y' = y + b_2
\]

Rigid
\[
x' = x \cos \theta - y \sin \theta + b_1 \\
y' = x \sin \theta + y \cos \theta + b_2
\]

Affine
\[
x' = a_1 x + a_2 y + b_1 \\
y' = a_3 x + a_4 y + b_2
\]

Projective
\[
x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1} \\
y' = \frac{a_3 x + a_4 y + b_1}{c_1 x + c_2 y + 1}
\]

Bi-quadratic
\[
x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 x y \\
y' = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} y^2 + a_{12} x y
\]

Bi-Linear
\[
x' = a_1 + a_2 x + a_3 y + a_4 x y \\
y' = a_5 + a_6 x + a_7 y + a_8 x y
\]

Pseudo-Perspective
\[
x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 x y \\
y' = a_6 + a_7 x + a_8 y + a_4 x y + a_5 y^2
\]
Displacement Models (contd)

• Translation
  – simple
  – used in block matching
  – no zoom, no rotation, no pan and tilt

• Rigid
  – rotation and translation
  – no zoom, no pan and tilt
Displacement Models (contd)

- Affine
  - rotation about optical axis only
  - cannot capture pan and tilt
  - orthographic projection

- Projective
  - exact eight parameters (3 rotations, 3 translations and 2 scalings)
  - difficult to estimate (due to denominator terms)
Displacement Models (contd)

• Biquadratic
  – obtained by second order Taylor series
  – 12 parameters
• Bilinear
  – obtained from biquadratic model by removing square terms
  – most widely used
  – not related to any physical 3D motion
• Pseudo-perspective
  – obtained by removing two square terms and constraining four remaining to 2 degrees of freedom
Spatial Transformations

- Translation
- Rotation
- Shear
- Rigid
- Affine
Instantaneous Velocity Model
3-D Rigid Motion

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = \begin{bmatrix}
1 & -\alpha & \beta \\
\alpha & 1 & -\gamma \\
-\beta & \gamma & 1
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

\[
\begin{bmatrix}
X' - X \\
Y' - Y \\
Z' - Z
\end{bmatrix} = \begin{bmatrix}
0 & -\alpha & \beta \\
\alpha & 0 & -\gamma \\
-\beta & \gamma & 0
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{bmatrix} = \begin{bmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} = \left(\begin{bmatrix}
0 & -\alpha & \beta \\
\alpha & 0 & -\gamma \\
-\beta & \gamma & 0
\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]
3-D Rigid Motion

\[
\begin{align*}
\dot{X} &= \Omega_{2}Z - \Omega_{3}Y + V_1 \\
\dot{Y} &= \Omega_{3}X - \Omega_{1}Z + V_2 \\
\dot{Z} &= \Omega_{1}Y - \Omega_{2}X + V_3 \\
\dot{X} &= \Omega \times X + V
\end{align*}
\]

Cross Product

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

\[
\Omega = \begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{bmatrix}
\]

\[
\Omega \times X = \begin{bmatrix}
i & j & k \\
\Omega_1 & \Omega_2 & \Omega_3 \\
X & Y & Z
\end{bmatrix} = \begin{bmatrix}
-\Omega_3 Y + \Omega_2 Z \\
\Omega_3 X - \Omega_1 Z \\
- X \Omega_2 + Y \Omega_1
\end{bmatrix}
\]
Orthographic Projection

\[ \dot{X} = \Omega_2 Z - \Omega_3 Y + V_1 \]
\[ \dot{Y} = \Omega_3 X - \Omega_1 Z + V_2 \]
\[ \dot{Z} = \Omega_1 Y - \Omega_2 X + V_3 \]

\[ u = \dot{x} = \Omega_2 Z - \Omega_3 y + V_1 \]
\[ v = \dot{y} = \Omega_3 x - \Omega_1 Z + V_2 \]

(u,v) is optical flow
Plane+orthographic (Affine)

\[ Z = a + bX + cY \]

\[ u = V_1 + \Omega_2 Z - \Omega_3 y \]
\[ v = V_2 + \Omega_3 x - \Omega_1 Z \]
\[ u = b_1 + a_1 x + a_2 y \]
\[ v = b_2 + a_3 x + a_4 y \]

Home work

\[ u = Ax + b \]

\[ b_1 = V_1 + a\Omega_2 \]
\[ a_1 = b\Omega_2 \]
\[ a_2 = c\Omega_2 - \Omega_3 \]
\[ b_2 = V_2 - a\Omega_1 \]
\[ a_3 = \Omega_3 - b\Omega_1 \]
\[ a_4 = -c\Omega_1 \]
Perspective Projection (arbitrary flow)

\[
x = \frac{fX}{Z}
\]

\[
y = \frac{fY}{Z}
\]

\[
\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1
\]

\[
\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2
\]

\[
\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3
\]

\[
\frac{\dot{x}}{Z} = \frac{\Omega_2}{f} y + \frac{V_1}{Z}
\]

\[
\frac{\dot{y}}{Z} = \frac{\Omega_3}{f} x - \frac{\Omega_1}{f} + \frac{V_2}{Z}
\]

\[
\frac{\dot{z}}{Z} = \frac{\Omega_1}{f} y - \frac{\Omega_2}{f} x + \frac{V_3}{Z}
\]

\[
u = \frac{fZ\dot{X} - fX\dot{Z}}{Z^2} = f \frac{\dot{X}}{Z} - x \frac{\dot{Z}}{Z}
\]

\[
u = \frac{fZ\dot{Y} - fY\dot{Z}}{Z^2} = f \frac{\dot{Y}}{Z} - y \frac{\dot{Z}}{Z}
\]

\[
u = f \left( \frac{V_1}{Z} + \Omega_2 \right) - \frac{V_3}{Z} x - \Omega_3 y - \frac{\Omega_1}{f} xy + \frac{\Omega_2}{f} x^2
\]

\[
u = f \left( \frac{V_2}{Z} - \Omega_1 \right) + \Omega_3 x - \frac{V_3}{Z} y + \frac{\Omega_2}{f} xy - \frac{\Omega_1}{f} y^2
\]
Plane+Perspective (pseudo perspective)

\[ u = f \left( \frac{V_1}{Z} + \Omega_2 \right) - \frac{V_3}{Z} x - \Omega_3 y - \frac{\Omega_1}{f} xy + \frac{\Omega_2}{f} x^2 \]

\[ v = f \left( \frac{V_2}{Z} - \Omega_1 \right) + \Omega_3 x - \frac{V_3}{Z} y + \frac{\Omega_2}{f} xy - \frac{\Omega_1}{f} y^2 \]

\[ Z = a + bX + cY \]

\[ \begin{vmatrix} 1 & 1 & b & c \\ \frac{1}{Z} & \frac{1}{a} & x & - \frac{1}{a} y \end{vmatrix} \]

\[ u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy \]

\[ v = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2 \]