Transformations Between Two Images

- Translation
- Rotation
- Rigid
- Similarity (scaled rotation)
- Affine
- Projective
- Pseudo Perspective
- Bi-linear
Fundamental Matrix

Lecture-12
Applications

• Stereo
• Structure from Motion
• View Invariant Action Recognition
• ..
Stereo Pairs and Depth Maps (from Szeliski’s book)
Image Rectification For Stereo
Photosynth: Structure From Motion
Fundamental Matrix

• Longuet Higgins (1981)
• Hartley (1992)
• Faugeras (1992)
• Zhang (1995)
Fundamental Matrix Song

http://www.youtube.com/watch?v=DgGV3l82NTk
Preliminaries

- Linear Independence
- Rank of a Matrix
- Matrix Norm
- Singular Value Decomposition
- Vector Cross product to Matrix Multiplication
- RANSAC
A finite subset of \( n \) vectors, \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \), from the vector space \( V \), is **linearly dependent** if and only if there exists a set of \( n \) scalars, \( a_1, a_2, \ldots, a_n \), not all zero, such that

\[
a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n = \mathbf{0}.
\]
Rank of a Matrix

• The **column rank** of a matrix $A$ is the maximum number of linearly independent column vectors of $A$.
• The **row rank** of a matrix $A$ is the maximum number of linearly independent row vectors of $A$.
• The column rank of $A$ is the dimension of the column space of $A$.
• The row rank of $A$ is the dimension of the row space of $A$. 
Example (Row Echelon)

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} R_2 \to 2r_1+r_2 \]

Rank is 2
Singular Value Decomposition (SVD)

Theorem: Any $m$ by $n$ matrix $A$, for which $m \geq n$, can be written as

$$A = O_1 \Sigma O_2$$

where $O_1$ and $O_2$ are orthogonal, $\Sigma$ is diagonal, and

$$O_1^T O_1 = O_2^T O_2 = I$$
Matrix Norm

L1 matrix norm is maximum of absolute column sum.

\[ \|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|, \]
Vector Cross Product to Matrix-vector multiplication

\[ A \times B = \begin{bmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \begin{bmatrix} -A_zB_y + A_yB_z \\ A_zB_x - A_xB_z \\ -B_xA_y + B_yA_x \end{bmatrix} \]

\[ A \times B = S.B = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} -A_zB_y + A_yB_z \\ A_zB_x - A_xB_z \\ -B_xA_y + B_yA_x \end{bmatrix} \]
RANSAC: Random Sampling and Consensus
RANSAC Song

http://www.youtube.com/watch?v=1YNjMxxXO-E&feature=relmfu
How to Fit A Line?

$$y = mx + c$$
How to Fit A Line?

- Least squares Fit (over constraint)
- RANSAC (constraint)
- Hough Transform (under constraint)
Least Squares Fit

- Standard linear solution to estimating unknowns.
  - If we know which points belong to which line
  - Or if there is only one line

\[ y = mx + c = f(x, m, c) \]

Minimize \( E = \sum_{i} [y_i - f(x_i, m, c)]^2 \)

Take derivative wrt \( m \) and \( c \) set to 0
Line Fitting

\[ y = mx + c \]

\[ y_1 = mx_1 + c \]
\[ y_2 = mx_2 + c \]
\[ \vdots \]
\[ y_n = mx_n + c \]

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n \\
\end{bmatrix}_{B} =
\begin{bmatrix}
  x_1 & 1 \\
  x_2 & 1 \\
  \vdots & \vdots \\
  x_n & 1 \\
\end{bmatrix}_{A}
\begin{bmatrix}
  m \\
  c \\
\end{bmatrix}_{D}
\Rightarrow B = AD
\]

\[
A^T B = A^T AD
\]

\[
 \left( A^T A \right)^{-1} A^T B = \left( A^T A \right)^{-1} (A^T A) D
\]

\[
D = \left( A^T A \right)^{-1} A^T B
\]
RANSAC: Random Sampling and Consensus

1. Randomly select two points to fit a line

2. Find the error between the estimated solution and all other points. If the error is less than tolerance, then quit, else go to step (1).
Derivation of Fundamental Matrix
Epipolar Geometry

- Defined for two static cameras
Epipolar Geometry

Epipolar line: set of world points that project to the same point in left image, when projected to right image forms a line.

Epipolar plane: plane defined by the camera centers and world point.

Epipole: intersection of image plane with line connecting camera centers. Image of a left camera center in the right, and vice versa.
Essential Matrix

Coplanarity constraint between vectors \((P_l - T), T, P_l\).

\[
(P_l - T)^T T \times P_l = 0
\]

\[
P_r = R(P_l - T)
\]

\[
P_r^T RT \times P_l = 0
\]
Essential Matrix

\[ P_r^T R T \times P_l = 0 \]

\[ P_r^T R S P_l = 0 \]

\[ P_r^T E P_l = 0 \]

essential matrix

\[ E = R S \]
Fundamental Matrix

Apply Camera model

\[
\begin{align*}
M_l^{-1} x_l &= P_l \\
M_r^{-1} x_r &= P_r \\
x_r^T M_r^{-T} &= P_r^T
\end{align*}
\]

\[
\begin{align*}
x_l &= M_l P_l \\
x_r &= M_r P_r \\
P_r^T E P_l &= 0
\end{align*}
\]

\[
x_r^T M_r^{-T} E M_l^{-1} x_l = 0 \\
x_r^T (M_r^{-T} E M_l^{-1}) x_l = 0 \\
x_r^T F x_l = 0
\]

fundamental matrix
Fundamental Matrix

\[ X'^T F X = x'^T \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix} x = 0 \]

• Given a point in left camera \( x \), epipolar line in right camera is: \( u_r = F x \)
Fundamental Matrix

- 3x3 matrix with 9 components
- Rank 2 matrix (due to $S$)
- 7 degrees of freedom
- Given a point in left camera $x$, epipolar line in right camera is: $u_r = Fx$
Fundamental Matrix

- Longuet Higgins (1981)
- Hartley (1992)
- Faugeras (1992)
- Zhang (1995)
Fundamental Matrix

- Fundamental matrix captures the relationship between the corresponding points in two views.

\[
\begin{bmatrix}
  x_i \\
  y_i \\
  1
\end{bmatrix}^T
\begin{bmatrix}
  f_{11} & f_{12} & f_{13} \\
  f_{21} & f_{22} & f_{23} \\
  f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
  x'_i \\
  y'_i \\
  1
\end{bmatrix} = 0,
\]

\[
\begin{bmatrix}
  x_i \\
  y_i \\
  1
\end{bmatrix}^T
\begin{bmatrix}
  f_{11}x'_i + f_{12}y'_i + f_{13} \\
  f_{21}x'_i + f_{22}y'_i + f_{23} \\
  f_{31}x'_i + f_{32}y'_i + f_{33}
\end{bmatrix} = 0,
\]
Fundamental Matrix

\[ x_i(f_{11}x' + f_{12}y' + f_{13}) + y_i(f_{21}x' + f_{22}y' + f_{23}) + (f_{31}x' + f_{32}y' + f_{33}) = 0 \]

\[ x_i x' f_{11} + x_i y' f_{12} + x_i f_{13} + y_i x' f_{21} + x' y' f_{22} + y_i f_{23} + x' f_{31} + y_i f_{32} + f_{33} = 0 \]

One equation for one point correspondence

\[
\begin{bmatrix}
x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 & x_1 & y_1 & 1 \\
x'_2 x_2 & x'_2 y_2 & x'_2 & y'_2 x_2 & y'_2 & x_2 & y_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n & x_n & y_n & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7 \\
f_8 \\
f_9
\end{bmatrix} = 0
\]

\[ Mf = 0 \]

M is 9 by \( n \) matrix

\[ f = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{21} & f_{22} & f_{23} & f_{31} & f_{32} & f_{33}
\end{bmatrix} \]

To solve the equation, the rank(M) must be 8.
Computation of Fundamental Matrix
Normalized 8-point algorithm (Hartley)

Objective:
Compute fundamental matrix $F$ such that $x'_iFx_i = 0$

Algorithm
Normalize the image $\hat{x}_i = Tx_i$, $\hat{x}'_i = T'x'_i$

Find centroid of points in each image, determine the range, and normalize all points between 0 and 1

Linear solution
determining the eigen vector corresponding to the smallest eigen value of $A$,

$$Af = \begin{bmatrix} \hat{x}'_1\hat{x}_1 & \hat{x}'_1\hat{y}_1 & \hat{x}'_1 & \hat{y}'_1\hat{x}_1 & \hat{y}'_1\hat{y}_1 & \hat{y}'_1 & \hat{x}_1 & \hat{y}_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} f = 0$$
Normalized 8-point algorithm (Hartley)

Construct
\[
\hat{F} = \begin{bmatrix}
  f_1 & f_2 & f_3 \\
  f_4 & f_5 & f_6 \\
  f_7 & f_8 & f_9 \\
\end{bmatrix}
\]

Normalize
\[
\hat{F} = \hat{F} / \| \hat{F} \|
\]

Constraint enforcement SVD decomposition
\[
\hat{F} = U \begin{bmatrix}
  \sigma_1 & 0 & 0 \\
  0 & \sigma_2 & 0 \\
  0 & 0 & \sigma_3 \\
\end{bmatrix} V'
\]

(\(\sigma_1 \geq \sigma_2 \geq \sigma_3\))

Rank enforcement
\[
\tilde{F} = U \begin{bmatrix}
  \sigma_1 & 0 & 0 \\
  0 & \sigma_2 & 0 \\
  0 & 0 & 0 \\
\end{bmatrix} V'
\]

(\(\sigma_3 = 0\))

De-normalization:
\[
F = T'^T \tilde{F} T
\]

L1 matrix norm is maximum of absolute column sum.
\[
\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|
\]
Robust Fundamental Matrix Estimation (by Zhang)

- Uniformly divide the image into 8×8 grid.
- Randomly select 8 grid cells and pick one pair of corresponding points from each grid cell, then use Hartley’s 8-point algorithm to compute Fundamental Matrix $F_i$.
- For each $F_i$, compute the median of the squared residuals $R_i$.
  - $R_i = \text{median}_k [d(p_{1k}, F_i p_{2k}) + d(p_{2k}, F'_i p_{1k})]$
- Select the best $F_i$ according to $R_i$.
- Determine outliers if $R_k > Th$.
- Using the remaining points compute the fundamental Matrix $F$ by weighted least square method.
Epi-polar Lines
Epi-polar lines